Matrix Algebra 2

:Vector Space

- The set of matrices $A \in \mathbb{R}^{mxn}$ endowed with the operations of matrix addition and scalar multiplication (as we have defined these operations) provides a special case of an important mathematical object called a *vector space* (over $\mathbb{R}$).
- Intuitively, a vector space is a collection of objects that is closed under linear combinations. That is (i) forming linear combinations (l.c.) makes sense, and (ii) forming linear combinations always leads to a vector in the collection.
- $\mathbb{R}^m$ is a vector space and we call its members *vectors*.

In what follows, we'll denote a vector space by V.

:Subspace (or linear manifold)

Definition: A set of vectors $S \subset V$ is called a *subspace* if $\forall x_1, x_2 \in S$ and $\forall c_1, c_2 \in \mathbb{R}$ then $c_1 x_1 + c_2 x_2 \in S$.

Rks:

- Because $S \subset V$, it inherits the property that l.c. makes sense.
- In words, S is a subspace if it is closed under l.c.
- A subspace is also a vector space.

Ex: Let $V = \mathbb{R}^m, S = \{X \in \mathbb{R}^m : X' = (x, 0, \cdots, 0)\}$. Then S is a subspace, since

$$c_1 X_1 + c_2 X_2 = \begin{bmatrix} c_1 x_1 + c_2 x_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in S$$

:Linear Span

Definition: Let $X = \{X_1, \cdots X_K\} \in V$. The *linear span* of X, denoted $Sp(X)$, is given by

$$Sp(X) = \{y \in V : y = \sum_{i=1}^{K} c_i X_i \text{ for some } c_i \in \mathbb{R}\}$$

Rks:

- In words, $Sp(X)$ is the set of all vectors that can be formed by taking l.c. of the members of X.
- $Sp(X)$ is a vector space. In fact, it is the smallest vector space that contains X.
- If X is a matrix, we write $Sp(X)$ for the span of its columns.

:Linear Dependence

Definition: A set of vectors $X = \{X_1, \cdots X_r\} \in V$ is said to be *linearly dependent* if there exist numbers $c_{1,} c_2, \cdots, c_r$ that are not all zero such that $c_1 X_1 + c_2 X_2 + \cdots + c_r X_r = 0$

Ex: Suppose $X_1^{'} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $X_2^{'} = \begin{bmatrix} 2 & 0 \end{bmatrix}$

Then $\{X_1, X_2\}$ is linearly dependent because $2X_1 - X_2 = 0$.

:Linear Independence

Definition: A set of vectors $X = \{X_1, \cdots X_r\} \in V$ is said to be *linearly independent* if it is not linearly dependent, i.e.

$$c_1 X_1 + c_2 X_2 + \cdots + c_r X_r = 0 \quad \Leftrightarrow$$

$$c_1 = c_2 = \cdots = c_r = 0$$

Note that here independence is algebraic, not statistical.

:Rank

Definition: The rank of a matrix A, denoted $rk(A)$ or $\rho(A)$ is its maximal number of linearly independent columns.

Rk: The rank also equals the maximal number of linearly independent rows.

:Basis

Definition: A linearly independent set X is a basis for the vector space V if $V = Sp(X)$.

: Dimension

Definition: The dimension of a vector space, $\dim(V)$, is the number of elements in the basis X.

Propositions:

1. Every vector space has a basis
2. The dimension of a vector space is unique, i.e. if $X = \{X_1, \cdots, X_k\}$ and $\widetilde{X} = \{\widetilde{X}_1, \cdots, \widetilde{X}_l\}$ are two choices for the basis, then $k = l$
3. Ever vector in V has a unique representation as a l.c. of the members of a fixed basis.

Proof: Suppose $Y = \sum a_i X_i$ and $Y = \sum b_i X_i$,

$$\Rightarrow \sum (a_i - b_i) X_i = 0 \Rightarrow a_i - b_i = 0$$

as $\{X_i\}$ is a basis, and therefore linearly independent.

:Geometry in $\mathbb{R}^m$ (note to self–Draw some pictures....)

Definition: The norm (length) of a vector $a \in \mathbb{R}^m$, denoted $\|a\|$, is given by $\|a\| = (a'a)^{1/2}$.

Rks:

- For $m = 1,\ \|a\| = |a|$
- For $m = 2,\ \|a\| = \sqrt{a_1^2 + a_2^2}$
- For $m = 3,\ \|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

So this definition generalizes the usual (Euclidean) notion of length to arbitrary dimensions.

Properties

- $\|ca\| = |c| \cdot \|a\| \ \ \forall c \in \mathbb{R}$
- $\|a + b\| \leq \|a\| + \|b\|$
- $\|a\| = 0 \text{ iff } a = 0$

Previous 3 properties hold for any norm. If norm comes from an inner product, we also get

- $|a'b| \leq \|a\| \cdot \|b\|$ (Cauchy-Schwartz inequality)

Definition: For a and $b \in \mathbb{R}^m$, the *angle* θ between them is defined by

$$\cos(\theta) = \frac{a'b}{\|a\| \cdot \|b\|}$$

Rks:

- In $\mathbb{R}^2$ and $\mathbb{R}^3$, this corresponds to our usual notion of angle
- By C-S, $\cos^2(\theta) \leq 1$

Definition: Two vectors a and $b \in \mathbb{R}^m$ are *orthogonal* to each other if $\cos(\theta) = 0 \Leftrightarrow a'b = 0$

Definition: Let V be a subspace. The vector a is normal to V, denoted $a \perp V$, if it's orthogonal to each $b \in V$.